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EMPIRICAL BAYES TESTS WITH $n^{-1+\epsilon}$ CONVERGENCE RATE IN CONTINUOUS ONE-PARAMETER EXPONENTIAL FAMILY

by

Shanti S. Gupta and Jianjun Li
Purdue University

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Department of Statistics
Purdue University
West Lafayette, IN USA

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Empirical Bayes Tests With $n^{-1+\varepsilon}$ Convergence Rate In Continuous One-Parameter Exponential Family

Shanti S. Gupta
Department of Statistics
Purdue University
W. Lafayette, IN 47907

and

Jianjun Li
Department of Statistics
Purdue University
W. Lafayette, IN 47907

Abstract

Empirical Bayes tests for testing $H_0 : \theta \leq \theta_0$ against $H_1 : \theta > \theta_0$ in the continuous one-parameter family with density $c(\theta)exp\{\theta x\}h(x)$, $\infty \leq \alpha < x < \beta \leq \infty$, are considered under the linear loss. Using the assumptions that $\int_\Omega |\theta| dG(\theta) < \infty$ and the critical point b_0 of a Bayes test falls in some known interval $[C_1, C_2]$, where $\alpha < C_1 < C_2 < \beta$, we show that, for any $0 < \varepsilon < 1$, the empirical Bayes tests can be constructed such that they have a convergence rate of order $o(n^{-1+\varepsilon})$, which generalizes the result of Liang (1999) from the positive (one-parameter) exponential family to any continuous one-parameter exponential family.

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§ 1 Introduction

Let X denote a random variable from the exponential family having density function

$$f(x|\theta) = c(\theta)exp\{\theta x\}h(x), \qquad -\infty \leq \alpha < x < \beta \leq +\infty, \tag{1.1}$$

where $h(x)$ is continuous, positive for $x \in (\alpha, \beta)$, θ is the natural parameter, distributed according to an unknown prior distribution G on the parameter space $\Omega = \{\theta : c(\theta) > 0\}$.

We consider the problem of testing the hypotheses $H_0 : \theta \leq \theta_0$ verses $H_1 : \theta > \theta_0$, where $\theta_0 \in \Omega$.

Let $a = i$ be the action in favor of H_i. For the parameter θ and action a, we use the loss function

$$l(\theta, a) = a(\theta_0 - \theta)I_{[\theta \leq \theta_0]} + (1 - a)(\theta - \theta_0)I_{[\theta > \theta_0]}, \tag{1.2}$$

where $I_{[.]}$ is the indicator function, which equals 1 or 0 if the statement inside [ ] is true or not. Assume that $\int_\Omega |\theta| dG(\theta) < \infty$. Define

$$\alpha_G(x) = \int_\Omega c(\theta)e^{\theta x} dG(\theta),$$

and

$$\psi_G(x) = \int_\Omega \theta c(\theta)e^{\theta x} dG(\theta).$$

By Fubini Theorem,

$$\begin{aligned} & \int_\alpha^\beta h(x)\alpha_G(x)dx \qquad (1.3) \\ = & \int_\alpha^\beta h(x) \int_\Omega c(\theta)e^{\theta x} dG(\theta)dx \\ = & \int_\Omega \int_\alpha^\beta c(\theta)e^{\theta x} h(x)dxdG(\theta) = 1 \end{aligned}$$

and

$$\begin{aligned} & \int_\alpha^\beta h(x)|\psi_G(x)|dx \qquad (1.4) \\ \leq & \int_\alpha^\beta h(x) \int_\Omega |\theta| c(\theta)e^{\theta x} dG(\theta)dx \\ = & \int_\Omega |\theta| \int_\alpha^\beta c(\theta)e^{\theta x} dxdG(\theta) \\ = & \int_\Omega |\theta| dG(\theta) < \infty. \end{aligned}$$

Let $W(x) = \theta_0\alpha_G(x) - \psi(x)$. Then $W(x)$ is a continuous function. By (1.3) and (1.4),

$$\begin{aligned} & \int_\alpha^\beta |W(x)|h(x)dx \\ < & \int_\alpha^\beta [|\theta_0|\alpha_G(x) + |\psi_G(x)|]h(x)dx < \infty. \end{aligned} \tag{1.5}$$

A test $\delta(x)$ is defined to be a measurable mapping from (α, β) into $[0,1]$ so that $\delta(x) = P\{$ accepting $H_1|X = x\}$, i.e., $\delta(x)$ is the probability of accepting H_1 when $X = x$ is observed.

Let $R(G,\delta)$ denote the Bayes risk of the test δ when G is the prior distribution. Then Bayes risk $R(G,\delta)$ can be expressed as

$$\begin{aligned} R(G,\delta) &= C_G + \int_\Omega \int_\alpha^\beta \delta(x)(\theta_0 - \theta)c(\theta)e^{\theta x}h(x)dxdG(\theta) \qquad (1.6) \\ &= C_G + \int_\alpha^\beta \delta(x)[\int_\Omega (\theta_0 - \theta)c(\theta)e^{\theta x}dG(\theta)]h(x)dx \\ &= C_G + \int_\alpha^\beta \delta(x)[\theta_0 \alpha_G(x) - \psi_G(x)]h(x)dx \\ &= C_G + \int_\alpha^\beta \delta(x)W(x)h(x)dx \\ &= C_G + \int_\alpha^\beta \delta(x)[\theta_0 - \phi_G(x)]\alpha_G(x)h(x)dx, \end{aligned}$$

where

$$C_G = \int_\Omega (\theta - \theta_0)I_{[\theta > \theta_0]}dG(\theta),$$

and

$$\phi_G(x) = E[\theta|X = x] = \frac{\psi_G(x)}{\alpha_G(x)}.$$

Here, $\phi_G(x)$ is the posterior mean of θ given $X = x$. $\phi_G(x)$ is continuous and increasing in x.

From (1.6), we see that a Bayes test δ_G is determined by

$$\delta_G(x) = \begin{cases} 1 & \text{if} \quad W(x) \le 0 \\ 0 & \text{if} \quad W(x) > 0 \end{cases} \qquad (1.7)$$

$$= \begin{cases} 1 & \text{if} \quad \phi_G(x) \ge \theta_0 \\ 0 & \text{if} \quad \phi_G(x) < \theta_0. \end{cases} \qquad (1.8)$$

The minimum Bayes risk is

$$R(G,\delta_G) = C_G + \int_\alpha^\beta \delta_G(x)W(x)h(x)dx. \qquad (1.9)$$

To exclude trivial cases, we assume that

$$\lim_{x\to\alpha} \phi_G(x) < \theta_0 < \lim_{x\to\beta} \phi_G(x). \qquad (1.10)$$

From (1.10), we get that $\phi_G(x)$ is stictly increasing and there exists the unique point b_0 (critical value) such that $\phi_G(b_0) = \theta_0$. $\phi_G(x) < \theta_0$ for $x < b_0$, and $\phi_G(x) > \theta_0$ for $x > b_0$. Therefore, the Bayes test δ_G can be represented as

$$\delta_G(x) = \begin{cases} 1 & \text{if} \quad x \ge b_0 \\ 0 & \text{if} \quad x < b_0. \end{cases}$$

Furthermore we assume that there exist two known constants C_1, C_2, $\alpha < C_1 < C_2 < \beta$ such that

$$C_1 \leq b_0 \leq C_2. \tag{1.11}$$

We will deal with this testing problem via the empirical Bayes approach. The empirical Bayes approach was introduced first by Robbins (1956, 1964). Let $X_1, X_2, \cdots, X_n$ denote the observations from n independent past experiences. Denote $\widetilde{X_n} = (X_1, X_2, \cdots, X_n)$. Let X be the present observation. An empirical Bayes test $\delta_n(X, \widetilde{X_n})$ is defined to be the probability of accepting H_1 when X and $\widetilde{X_n}$ are observed. Let $R(G, \delta_n|\widetilde{X_n})$ denote the Bayes risk of δ_n conditioning on $\widetilde{X_n}$ and $R(G, \delta) = E[R(G, \delta|\widetilde{X_n})]$ the overall (unconditional) Bayes risk of δ_n.

Since $R(G, \delta_G)$ is the minimum Bayes risk, $R(G, \delta_n|\widetilde{X_n}) - R(G, \delta_G) \geq 0$ for all $\widetilde{X_n}$ and for all n. Thus, the regret $R(G, \delta_G) - R(G, \delta_G) \geq 0$ for all n. The nonnegative regret $R(G, \delta_n) - R(G, \delta_G)$ is often used as a measure of performence of the empirical Bayes test of δ_n.

Johns and Van Ryzin (1972) constructed empirical Bayes tests for the continuous one-parameter exponential family and studied the rate of convergence for their associated regrets. Van Houwelingen (1976) improved Johns and Van Ryzin's result by using the monotonicity of the testing problem and showed that the empirical Bayes tests there have a convergence rate of order $O(n^{-2r/(2r+3)} log^2(n))$, where $r \geq 1$ is an integer, associated with the moment condition that $\int_\Omega |\theta|^{r+1} dG(\theta) < \infty$ and the r-times differentiability condition on $m(x)$. Under the assumptions $\int_\Omega |\theta|^{r+1} dG(\theta) < \infty$, (1.11), and a few others, Karunamuni and Yang (1995) claimed that the empirical Bayes tests, constructed by them, achieve an exact rate of convergence of order $O(n^{-2r/(2r+3)})$. Note that if r is small, these rates are still slow. Recently, Liang (1999) investigated the empirical Bayes test for the positive exponential family and much improved the previous results. His paper shows that the empirical Bayes tests there have a rate of convergence of order $O(n^{-s/(s+3)})$, where $s > 0$ is any prespecified number, under the (weaker) condition $\int_0^\infty \theta dG(\theta) < \infty$ and (1.11).

Our research interest on empirical Bayes tests is motivated by Liang (1999). Making full use of properties of $W(x)$, with the help of classical result about sum of i.i.d. random variables, under the assumption that $\int_\Omega |\theta| dG(\theta) < \infty$ and (1.11), we show that, for any $0 < \varepsilon < 1$, the empirical Bayes tests can be constructed such that they have a convergence rate of order $o(n^{-1+\varepsilon})$. Thus our result generalizes the result of Liang(1999) from the positive (one-parameter) exponential family to any continuous one-parameter exponential family.

The paper is organized as follows: §1 gives the introduction; §2 constructs a empirical Bayes test δ_n; §3 proves that the empirical Bayes test has a convergence rate of order $o(n^{-1+\varepsilon})$. §4 proves the lemmas stated in §3.

§ 2 Construction of Empirical Bayes Tests

We use the kernel method to construct the empirical Bayes tests. The idea here is similar to that of Stijnen (1985) and Liang (1999).

For any $0 < \varepsilon < 1$, take integer m such that $m\varepsilon > 4$. Suppose $[a, b]$ is a finite closed interval inside (α, β). For each $i = 0, 1$, let $K_i(y)$ be a Borel-measurable, bounded function vanishing outside the interal $[a, b]$ and for $K_0(y)$

$$\int_a^b y^j K_0(y)dy = \begin{cases} 1 & \text{if} \quad j = 0, \\ 0 & \text{if} \quad j = 1, 2, \cdots, m-1, \end{cases} \tag{2.1}$$

and for $K_1(y)$

$$\int_a^b y^j K_1(y)dy = \begin{cases} 0 & \text{if} \quad j = 0, 2, 3, \cdots, m, \\ 1 & \text{if} \quad j = 1. \end{cases} \tag{2.2}$$

We may let B_2 be a positive constant such that $|K_i(y)| \leq B_2$ for all $y \in [a, b]$ and $i = 0$ or 1.

Let $u = u(n) = n^{-\frac{\varepsilon}{4}}$. Then $u \to 0$ as $n \to \infty$. For any $x \in (\alpha, \beta)$, define

$$\alpha_n(x) = \frac{1}{nu} \sum_{j=1}^{n} K_0(\frac{X_j - x}{u})/h(X_j), \tag{2.3}$$

and

$$\psi_n(x) = \frac{1}{nu^2} \sum_{j=1}^{n} K_1(\frac{X_j - x}{u})/h(X_j). \tag{2.4}$$

Let $W_n(x) = \theta_0 \alpha_n(x) - \psi_n(x)$. We shall show later that $W_n(x)$ ia an asymptotically unbiased and consistent estimators of $W(x)$ (Lemma 3.2). Recalling that the critical value b_0 is inside $[C_1, C_2]$, then an empirical Bayes test $\delta_n(x, \widetilde{X}_n)$ can be proposed by

$$\delta_n = \begin{cases} 1 & \text{if} \quad (x > C_2) \text{ or } (C_1 \leq x \leq C_2 \text{ and } W_n(x) \leq 0), \\ 0 & \text{if} \quad (x < C_1) \text{ or } (C_1 \leq x \leq C_2 \text{ and } W_n(x) > 0). \end{cases} \tag{2.5}$$

The conditional Bayes risk of the empirical Bayes test δ_n is:

$$R(G, \delta_n | \widetilde{X}_n) = C_G + \int_\alpha^\beta \delta_n(x) W(x) h(x) dx. \tag{2.6}$$

Note that $W(x) \leq 0$ if $x \in [C_1, b_0]$; $W(x) \geq 0$ if $x \in [b_0, C_2]$. Then the conditional regret can be expressed as

$$\begin{aligned} R(G, \delta_n | \widetilde{X}_n) - R(G, \delta) &= \int_\alpha^\beta (\delta_n - \delta) W(x) h(x) dx \qquad (2.7) \\ &= \int_{C_1}^{b_0} I_{[W_n(x) \leq 0]} W(x) h(x) dx \\ &\quad + \int_{b_0}^{C_2} I_{[W_n(x) > 0]} |W(x)| h(x) dx \end{aligned}$$

and the unconditional regret becomes

$$R(G,\delta_n) - R(G,\delta) = \int_{C_1}^{b_0} P(W_n(x) \le 0) W(x) h(x) dx + \int_{b_0}^{C_2} P(W_n(x) > 0) |W(x)| h(x) dx. \qquad (2.8)$$

§3 Asymptotic Optimality of $\delta_n(x)$

In this section, we shall prove that $R(G,\delta_n) - R(G,\delta_G) = o(n^{-1+\varepsilon})$. The convergence rate of $R(G,\delta_n) - R(G,\delta_G)$ depends on the properties of $W(x)$ and $W_n(x)$. The more information about $W(x)$ and $W_n(x)$ (inlcuding $h(x)$) is used, the more accurate rate we will get. So firstly, we dig out a few properties of $W(x)$ and $W_n(x)$. That is a few lemmas, whose proofs are left to §4. Then we state two well-known facts. Following that, a desired convergence rate of $R(G,\delta_n) - R(G,\delta_G)$ is given as a theorem.

The first lemma is concerned about $W(x)$, which gives us a solution to deal with the case that $W(x)$ is small, but not small enough.

Lemma 3.1 *For any $\eta > 0$, define $l(\eta) = \int_{C_1}^{C_2} I_{[|W(x)| \le \eta]} dx$, the Lebesgue measure of $\{x : |W(x)| \le \eta\} \cap [C_1, C_2]$. Then there exists an $\eta_0 > 0$ and some positive constant $B_3 > 0$ such that, for any $\eta \le \eta_0$,*

$$l(\eta) \le B_3 \eta. \qquad (3.1)$$

Next we consider $W_n(x)$. We have two lemmas, which are direct results of computations. Note that

$$W_n(x) = \theta_0 \alpha_n(x) - \psi_n(x) = \frac{1}{n} \sum_{j=1}^{n} V(X_j, x, n), \qquad (3.2)$$

where

$$V(X_j, x, n) = \frac{\theta_0}{u} \times \frac{K_0(\frac{X_j - x}{u})}{h(X_j)} - \frac{1}{u^2} \times \frac{K_1(\frac{X_j - x}{u})}{h(X_j)}. \qquad (3.3)$$

Let $\overline{W}(x,n) = E[V(X_j,x,n)]$ and $Z_{jn} = V(X_j,x,n) - \overline{W}(x,n)$. Then we have

Lemma 3.2 *$\overline{W}(x,n)$ can be expressed as*

$$\overline{W}(x,n) = W(x) + u^m W(x,n),$$

where $W(x,n)$ is some function such that $|W(x,n)| \le B_4$ for all $x \in [C_1, C_2]$ and $n > N_1$, and where B_4 is some positive number and N_1 is some integer.

Also, we have

Lemma 3.3 *For any fixed n, Z_{jn} are i.i.d. and*

$$EZ_{jn} = 0, \qquad EZ_{jn}^2 = \frac{1}{u^3} D_2(x,n), \qquad E|Z_{jn}|^3 = \frac{1}{u^5} D_3(x,n),$$

where $D_2(x,n)$ and $D_3(x,n)$ are some functions such that $D_2(x,n) \le B_5$, $D_3(x,n) \le B_5$, $\frac{D_3(x,n)}{D_2(x,n)} \le B_5$ for all $x \in [C_1, C_2]$ and $n > N_1$, and where B_5 is some positive number and N_1 is same as in Lemma 3.2.

From Lemma 3.3, we see that

$$P(W_n(x) > 0) = P\Big(\frac{1}{\sqrt{nEZ_{jn}^2}} \sum_{j=1}^{n} Z_{jn} > -\sqrt{nu^3 D_2^{-1}(x,n)}\,\overline{W}(x,n)\Big), \tag{3.4}$$

and

$$P(W_n(x) \le 0) = P\Big(\frac{1}{\sqrt{nEZ_{jn}^2}} \sum_{j=1}^{n} Z_{jn} \le -\sqrt{nu^3 D_2^{-1}(x,n)}\,\overline{W}(x,n)\Big). \tag{3.5}$$

Comparing $W(x)$ and $\overline{W}(x,n)$, we get the following useful result:

Lemma 3.4 *There exists an integer $N_2(> N_1)$ such that, for any $n > N_2$ and $x \in [C_1, C_2]$,*

$$W(x) > \frac{1}{n} \Longrightarrow \overline{W}(x,n) \ge 0 \text{ and } \frac{W(x)}{\overline{W}(x,n)} \le 2, \tag{3.6}$$

and

$$W(x) < -\frac{1}{n} \Longrightarrow \overline{W}(x,n) \le 0 \text{ and } \Big|\frac{W(x)}{\overline{W}(x,n)}\Big| \le 2. \tag{3.7}$$

Lemma 3.4 allows us to replace $\overline{W}(x,n)$ with $W(x)$ in (3.4) and (3.5). That makes things a little easier since $W(x)$ does not depend on n and has a few good properties.

Next we state two general well-known results. One is about the non-uniform estimate of the distance between the distribution of a sum of i.i.d. random variables and the normal distribution; the other is about the normal quantile bounds.

Result A *Let $X_1, X_2, \cdots, X_n$ be i.i.d random variables, $EX_1 = 0$, $EX_1^2 = \sigma^2 > 0$, $E|X_1|^3 < \infty$. Then for all x*

$$|F_n(x) - \Psi(x)| \le A \frac{\rho}{\sqrt{n}(1+|x|)^3}. \tag{3.8}$$

Here $\Psi(x)$ is the c.d.f. of $N(0,1)$, $F_n(x)$ and ρ are given by

$$F_n(x) = P(\frac{1}{\sigma\sqrt{n}}\sum_{j=1}^{n} X_j \le x), \qquad \rho = \frac{E|X_1|^3}{\sigma^3}.$$

Remark Result A can be found in Petrov (1975, pp125 Theorem 14) or Michel (1981). Here A is independent of n. Michel proved $A < 30.54$.

Result B *Let $\Psi(x)$ be the c.d.f. of $N(0,1)$. Then for some constant $B_6 > 0$,*

$$x > 0 \Longrightarrow 1 - \Psi(x) = \int_x^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} dx \le \frac{B_6}{x} e^{-\frac{x^2}{2}}, \tag{3.9}$$

and

$$x < 0 \Longrightarrow \Psi(x) = \int_{-\infty}^{x} \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} dx \le \frac{B_6}{|x|} e^{-\frac{x^2}{2}}. \tag{3.10}$$

Combining (3.8), (3.9) and (3.10), we see that, for any fixed n, if $x > 0$,

$$\begin{aligned} & P(\frac{1}{\sigma\sqrt{n}}\sum_{j=1}^{n} X_j > x) \qquad (3.11) \\ = \ & 1 - F_n(x) \\ \le \ & 1 - \Psi(x) + A\frac{\rho}{\sqrt{n}(1+|x|)^3} \\ \le \ & \frac{B_6}{|x|} e^{-\frac{x^2}{2}} + \frac{A\rho}{\sqrt{n}|x|(1+|x|)^2}, \end{aligned}$$

if $x < 0$,

$$\begin{aligned} & P(\frac{1}{\sigma\sqrt{n}}\sum_{j=1}^{n} X_j \le x) \qquad (3.12) \\ = \ & F_n(x) \\ \le \ & \Psi(x) + A\frac{\rho}{\sqrt{n}(1+|x|)^3} \\ \le \ & \frac{B_6}{|x|} e^{-\frac{x^2}{2}} + \frac{A\rho}{\sqrt{n}|x|(1+|x|)^2}. \end{aligned}$$

Now, we prove our main result:

Theorem 3.5 *Let δ_n be the empirical Bayes test constructed in Section 2. Then we have, as $n \to \infty$,*

$$n^{1-\varepsilon}[R(G,\delta_n) - R(G,\delta)] \to 0.$$

Proof. For convinience, let $B_7 = \max_{C_1 \le x \le C_2} h(x)$ and $B_8 = \int_{C_1}^{C_2} h(x)dx < \infty$. From (2.8),

$$
\begin{aligned}
R(G,\delta_n) - R(G,\delta) &= \int_{C_1}^{b_0} P_n(W_n(x) \le 0) W(x) h(x) I_{[0<W(X)<\frac{1}{n}]} dx \qquad (3.13)\\
&+ \int_{b_0}^{C_2} P_n(W_n(x) > 0) |W(x)| h(x) I_{[-\frac{1}{n}<W(x)<0]} dx \\
&+ \int_{C_1}^{b_0} P_n(W_n(x) \le 0) W(x) h(x) I_{[W(X)>\frac{1}{n}]} dx \\
&+ \int_{b_0}^{C_2} P_n(W_n(x) > 0) |W(x)| h(x) I_{[W(x)<-\frac{1}{n}]} dx \\
&\equiv I + II + III + IV.
\end{aligned}
$$

Part I and Part II are trivial. Since we have $|W(x)| \le \frac{1}{n}$ in both,

$$I \le \frac{1}{n} \int_{C_1}^{b_0} h(x)dx \le \frac{1}{n} B_8 \tag{3.14}$$

and

$$II \le \frac{1}{n} \int_{b_0}^{C_2} h(x)dx \le \frac{1}{n} B_8. \tag{3.15}$$

Part III and Part IV are a little more complicated. We treat Part III first. Using (3.6) and (3.12), we have

$$
\begin{aligned}
III &\le \int_{C_1}^{b_0} P\Big(\frac{1}{\sqrt{nEZ_{jn}^2}} \sum_{j=1}^{n} Z_{jn} \le -\sqrt{nu^3 D_2^{-1}(x,n)}\overline{W}(x,n)\Big) I_{[W(x)>\frac{1}{n}]} W(x) h(x) dx \\
&\le \int_{C_1}^{b_0} P\Big(\frac{1}{\sqrt{nEZ_{jn}^2}} \sum_{j=1}^{n} Z_{jn} \le -\frac{1}{2}\sqrt{nu^3 D_2^{-1}(x,n)} W(x)\Big) I_{[W(x)>\frac{1}{n}]} W(x) h(x) dx \\
&\le \int_{C_1}^{b_0} \frac{2B_6 e^{-nu^3 D_2^{-1}(x,n) W^2(x)/8}}{\sqrt{nu^3 D_2^{-1}(x,n)} W(x)} I_{[W(x)>\frac{1}{n}]} W(x) h(x) dx \\
&+ \int_{C_1}^{b_0} \frac{2Au^{-5} D_3(x,n) W(x) h(x) I_{[W(x)>\frac{1}{n}]} dx}{[u^{-3} D_2(x,n)]^{3/2} \sqrt{n} \sqrt{nu^3 D_2^{-1}(x,n)} W(x) [1 + \frac{1}{2}\sqrt{nu^3 D_2^{-1}(x,n)} W(x)]^2} \\
&\le \int_{C_1}^{b_0} \frac{2B_6 e^{-nu^3 D_2^{-1}(x,n) W^2(x)/8}}{\sqrt{nu^3 D_2^{-1}(x,n)}} I_{[W(x)>\frac{1}{n}]} h(x) dx \\
&+ \int_{C_1}^{b_0} \frac{2AD_3(x,n)}{nu^2 D_2(x,n) [1 + \frac{1}{2}\sqrt{nu^3 D_2(x,n)} W(x)]^2} I_{[W(x)>\frac{1}{n}]} h(x) dx \\
&\le \int_{C_1}^{b_0} \frac{2B_6 e^{-nu^3 D_2^{-1}(x,n) W^2(x)/8}}{\sqrt{nu^3 D_2^{-1}(x,n)}} I_S h(x) dx
\end{aligned}
$$

$$
\begin{aligned}
&+\int_{C_1}^{b_0} \frac{2B_6}{\sqrt{nu^3 D_2^{-1}(x,n)}} I_{S^c} h(x)dx \\
&+\int_{C_1}^{b_0} \frac{2AD_3(x,n)}{nu^2 D_2(x,n)} h(x)dx \\
\equiv\ & V + VI + VII,
\end{aligned}
$$

where $S = \{x : e^{-nu^3 D_2^{-1}(x,n)W^2(x)/8} \leq \frac{1}{\sqrt{n}}\}$ and $S^c = \{x : e^{-nu^3 D_2^{-1}(x,n)W^2(x)/8} > \frac{1}{\sqrt{n}}\}$. Obviously,

$$
VII \leq \frac{2A}{nu^2} \int_{C_1}^{b_0} \frac{D_3(x,n)}{D_2(x,n)} h(x)dx \leq \frac{2A}{nu^2} B_5 B_8. \tag{3.16}
$$

By our definition of S,

$$
V \leq \frac{2B_6}{n\sqrt{u^3}} \int_{C_1}^{b_0} \sqrt{D_2(x,n)} h(x)dx \leq \frac{2}{n\sqrt{u^3}} \sqrt{B_5} B_6 B_8. \tag{3.17}
$$

As for Part VI, we have firstly that

$$
\begin{aligned}
e^{-nu^3 D_2^{-1}(x,n)W^2(x)/8} > \frac{1}{\sqrt{n}} &\implies W^2(x) < \frac{4D_2(x,n)\log n}{nu^3} \\
&\implies W^2(x) < \frac{4B_5 \log n}{nu^3} \\
&\implies |W(x)| < \sqrt{\frac{4B_5 \log n}{nu^3}}.
\end{aligned}
$$

So $I_{S^c} \leq I_{[|W(x)| \leq \eta]}$, where $\eta = \sqrt{\frac{4B_5 \log n}{nu^3}}$. Note that $\sqrt{\frac{4B_5 \log n}{nu^3}} \to 0$ as $n \to \infty$ since $u = n^{-\frac{\varepsilon}{4}}$. Let $N_3 > N_2 (> N_1)$ be an integer and such that for $n \geq N_3$, $\sqrt{\frac{4B_5 \log n}{nu^3}} < \eta_0$. From Lemma 3.1, when $n \geq N_3$,

$$
\begin{aligned}
VI &\leq \frac{2B_6}{\sqrt{nu^3}} \sqrt{B_5} B_7 \int_{C_1}^{b_0} I_{S^c} dx \qquad (3.18) \\
&\leq \frac{2B_6}{\sqrt{nu^3}} \sqrt{B_5} B_7 B_3 \sqrt{\frac{4B_5 \log n}{nu^3}} \\
&\leq \frac{\sqrt{\log n}}{nu^3} 4B_3 B_5 B_6 B_7.
\end{aligned}
$$

Combining (3.16), (3.17) and (3.18), we get that when $n \geq N_3$,

$$
III \leq \frac{2A}{nu^2} B_5 B_8 + \frac{2}{n\sqrt{u^3}} \sqrt{B_5} B_6 B_8 + \frac{\sqrt{\log n}}{nu^3} 4B_3 B_5 B_6 B_7. \tag{3.19}
$$

Now we deal with IV. Using (3.7) and (3.11), we get

$$
IV \leq \int_{b_0}^{C_2} P\Big(\frac{1}{\sqrt{nEZ_{jn}^2}} \sum_{j=1}^{n} Z_{jn} > -\sqrt{nu^3 D_2^{-1}(x,n)}\, \overline{W}(x,n)\Big) I_{[W(x) < -\frac{1}{n}]} |W(x)| h(x)dx
$$

$$
\begin{aligned}
&\le \int_{b_0}^{C_2} P\Big(\frac{1}{\sqrt{nEZ_{jn}^2}}\sum_{j=1}^{n} Z_{jn} > \frac{1}{2}\sqrt{nu^3 D_2^{-1}(x,n)}|W(x)|\Big) I_{[W(x)<-\frac{1}{n}]}|W(x)|h(x)dx \\
&\le \int_{b_0}^{C_2} \frac{2B_6 e^{-nu^3 D_2^{-1}(x,n)W^2(x)/8}}{\sqrt{nu^3 D_2^{-1}(x,n)}|W(x)|} I_{[W(x)<-\frac{1}{n}]}|W(x)|h(x)dx \\
&\quad + \int_{b_0}^{C_2} \frac{2AD_3(x,n)|W(x)|h(x)}{nu^2 D_2(x,n)|W(x)|[1+\frac{1}{2}\sqrt{nu^3 D_2^{-1}(x,n)}|W(x)|]^2} I_{[W(x)<-\frac{1}{n}]}dx \\
&\le \int_{b_0}^{C_2} \frac{2B_6 e^{-nu^3 D_2^{-1}(x,n)W^2(x)/8}}{\sqrt{nu^3 D_2^{-1}(x,n)}} I_S h(x)dx \\
&\quad + \int_{b_0}^{C_2} \frac{2B_6}{\sqrt{nu^3 D_2^{-1}(x,n)}} I_{S^c} h(x)dx \\
&\quad + \int_{b_0}^{C_2} \frac{2AD_3(x,n)}{nu^2 D_2(x,n)} h(x)dx \\
&\equiv VIII + IX + X.
\end{aligned}
$$

Recalling $S = \{x : e^{-nu^3 D_2^{-1}(x,n)W^2(x)/8} \le \frac{1}{\sqrt{n}}\}$ and $S^c = \{x : e^{-nu^3 D_2^{-1}(x,n)W^2(x)/8} > \frac{1}{\sqrt{n}}\}$, obviously,

$$
VIII \le \frac{2B_6}{n\sqrt{u^3}} \int_{b_0}^{C_2} \sqrt{D_2(x,n)}h(x)dx \le \frac{2B_6}{n\sqrt{u^3}}\sqrt{B_5}B_8. \tag{3.20}
$$

Similarly to (3.16),

$$
X \le \frac{2A}{nu^2}\int_{b_0}^{C_2} \frac{D_3(x,n)}{D_2(x,n)}h(x)dx \le \frac{2A}{nu^2}B_5B_8. \tag{3.21}
$$

As for Part IX, similarly to Part VI, we have

$$
\begin{aligned}
e^{-nu^3 D_2^{-1}(x,n)W^2(x)/8} > \frac{1}{\sqrt{n}} &\Longrightarrow W^2(x) < \frac{4D_2(x,n)\log n}{nu^3} \\
&\Longrightarrow W^2(x) < \frac{4B_5\log n}{nu^3} \\
&\Longrightarrow |W(x)| < \sqrt{\frac{4B_5\log n}{nu^3}}.
\end{aligned}
$$

When $n > N_3$, $\frac{\sqrt{4B_5\log n}}{\sqrt{nu^3}} \le \eta_0$ and $I_{S^c} \le I_{[|W(x)|\le \frac{\sqrt{4B_5\log n}}{\sqrt{nu^3}}]}$. From Lemma 3.1,

$$
\begin{aligned}
IX &\le \frac{2B_6}{\sqrt{nu^3}}\sqrt{B_5}B_7\int_{b_0}^{C_2} I_{S^c}dx \qquad (3.22)\\
&\le \frac{2B_6}{\sqrt{nu^3}}\sqrt{B_5}B_7B_3\frac{\sqrt{4B_5\log n}}{\sqrt{nu^3}} \\
&\le \frac{\sqrt{\log n}}{nu^3}4B_3B_5B_6B_7.
\end{aligned}
$$

Combining (3.20), (3.21) and (3.22), we get that when $n \geq N_3$,

$$IV \leq \frac{2}{n\sqrt{u^3}}\sqrt{B_5}B_6B_8 + \frac{2A}{nu^2}B_5B_8 + \frac{\sqrt{\log n}}{nu^3}4B_3B_5B_6B_7. \tag{3.23}$$

From (3.14), (3.15) (3.19) and (3.23), we get

$$R(G,\delta_n) - R(G,\delta) \leq \frac{4}{n\sqrt{u^3}}\sqrt{B_5}B_6B_8 + \frac{4A}{nu^2}B_5B_8 + \frac{\sqrt{\log n}}{nu^3}8B_3B_5B_6B_7]. \tag{3.24}$$

As $n \to \infty$,

$$\begin{cases} n^{1-\varepsilon}\frac{1}{n\sqrt{u^3}} = n^{1-\varepsilon}\frac{1}{nn^{-3\varepsilon/8}} = n^{-5\varepsilon/8} \to 0; \\ n^{1-\varepsilon}\frac{1}{u^2} = n^{1-\varepsilon}\frac{1}{nn^{-\varepsilon/2}} = n^{-\varepsilon/2} \to 0; \\ n^{1-\varepsilon}\frac{\sqrt{\log n}}{nu^3} = n^{1-\varepsilon}\frac{\sqrt{\log n}}{nn^{-3\varepsilon/4}} = n^{-\varepsilon/4}\sqrt{\log n} \to 0. \end{cases}$$

We get that $n^{1-\varepsilon}[R(G,\delta_n) - R(G,\delta)] \to 0$.
The proof is completed.

§ 4 Proofs of Lemmas

Proof of Lemma 3.1 Since $\alpha(x) < \infty$ for all x, $\alpha^{(l)}(x)$ exists for all x and all $l \geq 1$. Then $W^{(l)}(x)$ exists for all x and all $l \geq 1$, since $\psi(x) = \alpha'(x)$. These will be used in the proof of Lemma 3.2. Now, we need that

$$W'(x) = \theta_0 \int_\Omega \theta c(\theta)e^{\theta x}dG(\theta) - \int_\Omega \theta^2 c(\theta)e^{\theta x}dG(\theta). \tag{4.1}$$

First, we prove that

$$W'(b_0) < 0. \tag{4.2}$$

If $\psi_G(b_0) = 0$, then $W'(b_0) = -\int_\Omega \theta^2 c(\theta)e^{\theta b_0}dG(\theta) < 0$. If $\psi_G(b_0) > 0$, then

$$\frac{\psi'_G(b_0)}{\psi_G(b_0)} = \frac{\int_\Omega \theta^2 c(\theta)e^{\theta b_0}dG(\theta)}{\int_\Omega \theta c(\theta)e^{\theta b_0}dG(\theta)} > \frac{\int_\Omega \theta c(\theta)e^{\theta b_0}dG(\theta)}{\int_\Omega c(\theta)e^{\theta b_0}dG(\theta)} = \theta_0.$$

Thus $W'(b_0) = \psi_G(b_0)[\theta_0 - \frac{\psi'_G(b_0)}{\psi_G(b_0)}] < 0$. If $\psi_G(b_0) < 0$, then

$$\frac{\psi'_G(b_0)}{\psi_G(b_0)} = \frac{\int_\Omega \theta^2 c(\theta)e^{\theta b_0}dG(\theta)}{\int_\Omega \theta c(\theta)e^{\theta b_0}dG(\theta)} < \frac{\int_\Omega \theta c(\theta)e^{\theta b_0}dG(\theta)}{\int_\Omega c(\theta)e^{\theta b_0}dG(\theta)} = \theta_0.$$

Thus $W'(b_0) = \psi_G(b_0)[\theta_0 - \frac{\psi'_G(b_0)}{\psi_G(b_0)}] < 0$. Then (4.2) is proved.

From (4.1), we see that $W'(x)$ is continuous. Then we can find $b'_0 > 0$ such that, for any $x \in [b_0 - b'_0, b_0 + b'_0] \subset [C_1, C_2]$,

$$-W'(x) \geq \frac{1}{2}[-W'(b_0)] \equiv 2B_3^{-1}. \tag{4.3}$$

Note that $W(x) > 0$ for $x \in [C_1, b_0)$; $W(x) < 0$ for $x \in (b_0, C_2]$. Then

$$\eta_0 \equiv [\min_{C_1 \leq x \leq b_0 - b'_0} W(x)] \wedge [\min_{b_0 + b'_0 \leq x \leq C_2} W(x)] > 0,$$

where $a \wedge b = \min\{a, b\}$. For $\eta < \eta_0$, let

$$\eta_L = \{x : W(x) = \eta, C_1 \leq x \leq C_2\},$$

and

$$\eta_R = \{x : W(x) = -\eta, C_1 \leq x \leq C_2\}.$$

Since $\eta < \eta_0$, η_L and η_R are unique, and

$$[\eta_L, \eta_R] \subset [b_0 - b'_0, b_0 + b'_0].$$

Recall $W'(x) < 0$ for $x \in [b_0 - b'_0, b_0 + b'_0]$. Then $l(\eta) = \eta_R - \eta_L$. Using the slope formula and (4.3), we have

$$-\frac{-\eta - \eta}{\eta_R - \eta_L} \geq 2B_3^{-1}.$$

Thus

$$l(\eta) \leq B_3 \eta. \tag{4.4}$$

Proof of Lemma 3.2 Using Taylor's Theorem, (2.1) and (2.2), a straight-forword computation shows that

$$\begin{aligned}
& E[\frac{\theta_0 K_0(\frac{X_j - x}{u})}{uh(X_j)}] \qquad (4.5) \\
= & \int_\Omega \int_\alpha^\beta \frac{\theta_0 K_0(\frac{y-x}{u})}{uh(y)} c(\theta) e^{\theta y} h(y) dy dG(\theta) \\
= & \theta_0 \int_\Omega \int_a^b K_0(t) c(\theta) e^{\theta x} e^{\theta u t} dt dG(\theta) \\
= & \theta_0 \int_\Omega c(\theta) e^{\theta x} [\int_a^b K_0(t) e^{\theta u t} dt] dG(\theta) \\
= & \theta_0 \int_\Omega c(\theta) e^{\theta x} [1 + \frac{u^m}{m!} \theta^m \int_a^b K_0(t) t^m e^{\theta u t^*} dt] dG(\theta) \\
= & \theta_0 \int_\Omega c(\theta) e^{\theta x} dG(\theta) + u^m \int_\Omega \theta_0 \theta^m c(\theta) e^{\theta x} [\frac{1}{m!} \int_a^b K_0(t) t^m e^{\theta u t^*} dt] dG(\theta),
\end{aligned}$$

where $|t^*| \leq \max\{|a|, |b|\} \equiv c$. Also,

$$\begin{aligned}
& E[\frac{K_1(\frac{X_j-x}{u})}{u^2h(X_j)}] \qquad (4.6) \\
= & \int_\Omega \int_\alpha^\beta \frac{K_1(\frac{y-x}{u})}{u^2h(y)} c(\theta)e^{\theta y}h(y)dydG(\theta) \\
= & \frac{1}{u}\int_\Omega \int_a^b K_1(t)c(\theta)e^{\theta x}e^{\theta ut}dtdG(\theta) \\
= & \frac{1}{u}\int_\Omega c(\theta)e^{\theta x}[\int_a^b K_1(t)e^{\theta ut}dt]dG(\theta) \\
= & \frac{1}{u}\int_\Omega c(\theta)e^{\theta x}[u\theta + \frac{u^{m+1}}{(m+1)!}\theta^{m+1}\int_a^b K_1(t)t^{m+1}e^{\theta ut^{**}}dt]dG(\theta) \\
= & \int_\Omega \theta c(\theta)e^{\theta x}dG(\theta) + u^m \int_\Omega \theta^{m+1}c(\theta)e^{\theta x}[\frac{1}{(m+1)!}\int_a^b K_1(t)t^{m+1}e^{\theta ut^{**}}dt]dG(\theta),
\end{aligned}$$

where $|t^{**}| \leq \max\{|a|, |b|\} = c$. From (4.5) and (4.6), we get that

$$E[V(X_j, x, n)] = W(x) + u^m W(x, n),$$

where

$$\begin{aligned}
W(x,n) = & \theta_0 \int_\Omega \theta^m c(\theta)e^{\theta x}[\frac{1}{m!}\int_a^b K_0(t)t^m e^{\theta ut^*}dt]dG(\theta) \\
& - \int_\Omega \theta^{m+1}c(\theta)e^{\theta x}[\frac{1}{(m+1)!}\int_a^b K_1(t)t^{m+1}e^{\theta ut^{**}}dt]dG(\theta).
\end{aligned}$$

Choose $\zeta > 0$ such that $\alpha < C_1 - \zeta < C_2 + \zeta < \beta$. Then, we can find an N_1 such that for $n > N_1$,

$$uc \leq \zeta. \qquad (4.7)$$

Then

$$\begin{aligned}
|W(x,n)| \leq & B_2|\theta_0|c^m(b-a)\int_\Omega |\theta|^m c(\theta)e^{\theta x}(e^{\theta\zeta} + e^{-\theta\zeta})dG(\theta) \\
& + B_2c^{m+1}(b-a)\int_\Omega |\theta|^{m+1}c(\theta)e^{\theta x}(e^{\theta\zeta} + e^{-\theta\zeta})dG(\theta).
\end{aligned}$$

Let m_0 be an even number such that $m + 1 \leq m_0$. Then

$$\begin{aligned}
& \int_\Omega |\theta|^m c(\theta)e^{\theta x}(e^{\theta\zeta} + e^{-\theta\zeta})dG(\theta) \\
\leq & \int_{\Omega[|\theta|\leq 1]} c(\theta)e^{\theta x}(e^{\theta\zeta} + e^{-\theta\zeta})dG(\theta) + \int_{\Omega[|\theta|>1]} \theta^{m_0}c(\theta)e^{\theta x}(e^{\theta\zeta} + e^{-\theta\zeta})dG(\theta) \\
\leq & \int_\Omega c(\theta)e^{\theta x}(e^{\theta\zeta} + e^{-\theta\zeta})dG(\theta) + \int_\Omega \theta^{m_0}c(\theta)e^{\theta x}(e^{\theta\zeta} + e^{-\theta\zeta})dG(\theta) \\
= & \alpha_G(x+\zeta) + \alpha_G(x-\zeta) + \alpha_G^{(m_0)}(x+\zeta) + \alpha_G^{(m_0)}(x-\zeta).
\end{aligned}$$

Similarly

$$\int_\Omega |\theta|^{m+1}c(\theta)e^{\theta x}(e^{\theta\zeta}+e^{-\theta\zeta})dG(\theta) \le \alpha_G(x+\zeta)+\alpha_G(x-\zeta)+\alpha_G^{(m_0)}(x+\zeta)+\alpha_G^{(m_0)}(x-\zeta).$$

Since $\alpha_G^{(l)}$ is continuous, it follows that

$$B_9 = \max_{C_1-\zeta\le x\le C_2+\zeta} \alpha_G(x) < \infty,$$

and

$$B_{10} = \max_{C_1-\zeta\le x\le C_2+\zeta} \alpha_G^{(m_0)}(x) < \infty.$$

Thus

$$|W(x,n)| \le B_2|\theta_0|c^m(b-a)\times 2(B_9+B_{10}) + B_2c^{m+1}(b-a)\times 2(B_9+B_{10}) \equiv B_4 < \infty.$$

Proof of Lemma 3.3 Obviously, Z_{jn} are i.i.d. for fixed n. A few computations show that

$$\begin{aligned} EZ_{jn}^2 &= \frac{1}{u^4}\int_\Omega\int_\alpha^\beta [\theta_0 u\frac{K_0(\frac{y-x}{u})}{h(y)} - \frac{K_1(\frac{y-x}{u})}{h(y)} - u^2\overline{W}(x,n)]^2 c(\theta)e^{\theta y}h(y)dydG(\theta) \\ &= \frac{1}{u^3}\int_\Omega\int_a^b [\frac{\theta_0 uK_0(t)-K_1(t)}{h(x+ut)} - u^2\overline{W}(x,n)]^2 c(\theta)e^{\theta x}e^{\theta ut}h(x+ut)dtdG(\theta) \\ &= \frac{1}{u^3}D_2(x,u), \end{aligned}$$

where

$$D_2(x,u) = \int_\Omega\int_a^b [\frac{\theta_0 uK_0(t)-K_1(t)}{h(x+ut)} - u^2\overline{W}(x,n)]^2 c(\theta)e^{\theta x}e^{\theta ut}h(x+ut)dtdG(\theta).$$

Also,

$$\begin{aligned} E|Z_{jn}|^3 &= \frac{1}{u^6}\int_\Omega\int_\alpha^\beta |\theta_0 u\frac{K_0(\frac{y-x}{u})}{h(y)} - \frac{K_1(\frac{y-x}{u})}{h(y)} - u^2\overline{W}(x,n)|^3 c(\theta)e^{\theta y}h(y)dydG(\theta) \\ &= \frac{1}{u^5}\int_\Omega\int_a^b |\frac{\theta_0 uK_0(t)-K_1(t)}{h(x+ut)} - u^2\overline{W}(x,n)|^3 c(\theta)e^{\theta x}e^{\theta ut}h(x+ut)dtdG(\theta) \\ &= \frac{1}{u^5}D_3(x,u), \end{aligned}$$

where

$$D_3(x,u) = \int_\Omega\int_a^b |\frac{\theta_0 uK_0(t)-K_1(t)}{h(x+ut)} - u^2\overline{W}(x,n)|^3 c(\theta)e^{\theta x}e^{\theta ut}h(x+ut)dtdG(\theta).$$

When $n \geq N_1$, for any $x \in [C_1, C_2]$ and $t \in [a,b]$, $x + tu \in [C_1 - \zeta, C_2 + \zeta]$ and

$$\begin{aligned}
\overline{W}(x,n) &= W(x) + u^m W(x,n) \\
&\leq \max_{C_1 \leq x \leq C_2} W(x) + u^m B_4 \\
&\leq \max_{C_1 \leq x \leq C_2} W(x) + B_4 \\
&\equiv B_{11}.
\end{aligned}$$

Let $B_{12} = \max_{x \in [C_1-\zeta, C_2+\zeta]} \frac{1}{h(x)} < \infty$ and $B_{13} = \max_{x \in [C_1-\zeta, C_2+\zeta]} h(x) < \infty$. Then, for any $n \geq N_1$ and any $x \in [C_1, C_2]$,

$$\begin{aligned}
\left|\frac{\theta_0 u K_0(t) - K_1(t)}{h(x+ut)} - u^2 \overline{W}(x,n)\right| &\leq (|\theta_0| u B_2 + B_2) B_{12} + u^2 B_{11} \\
&\leq (|\theta_0| + 1) B_2 B_{12} + B_{11} \\
&\equiv B_{14},
\end{aligned}$$

and

$$\begin{aligned}
&\int_\Omega \int_a^b c(\theta) e^{\theta x} e^{\theta u t} h(x+ut) dt dG(\theta) \\
\leq\ & B_{13}(b-a) \int_\Omega c(\theta) e^{\theta x} (e^{\theta\zeta} + e^{-\theta\zeta}) dG(\theta) \\
\leq\ & 2B_{13}(b-a) \max_{x \in [C_1-\zeta, C_2+\zeta]} \alpha_G(x) \\
\equiv\ & B_{15}.
\end{aligned}$$

Therefore

$$D_2(x,n) \leq B_{14}^2 B_{15}, \qquad D_3(x,n) \leq B_{14}^3 B_{15}, \qquad \frac{D_3(x,n)}{D_2(x,n)} \leq B_{14}.$$

Letting $B_5 = \max\{B_{14}, B_{14}^2 B_{15}, B_{14}^3 B_{15}\} < \infty$, completes the proof.

Proof of Lemma 3.4 Noting that $u = n^{-\frac{\varepsilon}{4}}$ and $m\varepsilon > 4$, $nu^m = n^{-\frac{\varepsilon m - 4}{4}} \to 0$ as $n \to \infty$. Since $W(x,n) \leq B_4$ for all $x \in [C_1, C_2]$ if $n > N_1$, there exists an $N_2 (\geq N_1)$ such that $|nu^m W(x,n)| \leq \frac{1}{2}$ for all $x \in [C_1, C_2]$. If $W(x) > \frac{1}{n}$,

$$n\overline{W}(x,n) = n[W(x) + u^m W(x,n)] = nW(x) + nu^m W(x,n) > 1 - \frac{1}{2} = \frac{1}{2} > 0$$

and

$$\begin{aligned}
\frac{W(x)}{\overline{W}(x,n)} &= \frac{nW(x)}{nW(x) + nu^m W(x,n)} \\
&\leq \frac{nW(x) - 1 + 1}{nW(x) - 1 + \frac{1}{2}} \\
&\leq 2.
\end{aligned}$$

Then (3.6) is proved. (3.7) can be proved in a similar way.

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